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The application of Chebyshev polynomials to the solution of the nonprismatic Timoshenko beam vibration problem

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Abstract

Chebyshev series approximation is applied to solve the problem of vibration of the nonprismatic Timoshenko beam resting on a two-parameter elastic foundation. As a result, a system of equations (whose coefficients have a closed form) for calculating the coefficients of the sought solution is obtained. The method is used to solve the free vibration problem for simple supported and clamped–free nonprismatic beams. The results are compared with the results obtained by other authors. Also the nonprismatic beam stability problem is solved and the results are compared with those obtained for Euler beams. To demonstrate the method's applicability to more complex systems the problem of stability of a frame system is solved.

1. Introduction

The analysis of beam systems with variable material and geometric parameters has attracted much interest because of the need for rational and economic shaping of structures. The interest is reflected in the large number of papers devoted to this subject. Passing over the extensive literature on the dynamics of Euler beam systems (e.g. Refs. [1,2]), we shall limit ourselves to a review of works dealing with the Timoshenko beam. An analytical solution of the nonprismatic Timoshenko beam free vibration problem was presented by, among others, Huang [3] who reduced a system of two differential equations to one equation which he solved analytically. The solution is expressed by trigonometric and hyperbolic functions. Using the same method Bruch and Mitchell [4] solved the problem of free vibration of a cantilever beam with a mass element with nonnegligible rotatory inertia, placed at its end. Many of the published works on the Timoshenko beam vibration problem deal with the dynamic stability problem. The problem was investigated by, among others, Katsikadelis and Kounadis [5,6]. An analytical solution of the Beck column stability problem was presented in Ref. [5]. Kounadis derived equations for the problem of vibration of the Timoshenko beam subjected to a concentrated load and a uniformly distributed follower load [6]. The equations were derived in three independent ways by applying the principle of virtual work, Hamilton's principle and the equilibrium method. Sato applied the variational Hamilton principle to derive equations describing the vibration of the Timoshenko beam loaded with an axial force and a tangential force [7]. The transfer matrix method was

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applied to solve the problems of vibration and stability of the nonprismatic Timoshenko beam in a paper by Irie et al. [8]. The solution (sought in the form of power series) of the stability problem for variously supported nonprismatic beams was presented by Esmailzadeh and Ohadi in Ref. [9]. The nonprismatic beam vibration problem was solved using the finite element method by Klasztorny in Ref. [10] where a polynomial approximation was assumed for the shape functions on the basis of which rigidity and inertia matrices were then determined for finite Euler and Timoshenko beam elements. The Laplace transformation was applied to solve the problem in papers by Lueschen et al. [11] and Saito and Otomi [12]. Green functions were determined and compared for prismatic Euler and Timoshenko beams in Ref. [11]. In the case of the Euler beam, time-invariable axial loading of the beam was taken into account. The problems of vibration and stability of the Timoshenko beam with an added mass element (with non-negligible rotatory inertia), subjected to axial and tangential loads, were solved in Ref. [12]. A different approach to the nonprismatic beam vibration problem was employed by Tong et al. [13] who approximated the nonprismatic beam with a segmentally prismatic beam with stepwise variable parameters and using analytical solutions for the prismatic beam and continuity conditions solved the problem. Using distribution theory formalism Yavari et al. solved the problem of vibration of the beam with stepwise variable parameters [14]. The vibration of more complex systems was studied by, among others, Posiadała who in Ref. [15] solved the problem of transverse vibration of the Timoshenko beam with attachments such as springs, mass elements with a zero and nonzero moment of inertia, linear oscillators and additional supports, using the Lagrange equations and approximating the functions of displacements and angles of cross-section rotation with a series relative to the eigenfunctions obtained by solving the problem without attachments. Variational formalism with Lagrange multipliers was applied.

Most of the papers quoted above, i.e. Refs. [3–7,12,15], deal with prismatic beams. From the mathematical point of view, the beams are described by differential equations with constant coefficients. Whereas an analysis of nonprismatic beams leads to a system of equations with variable coefficients. Whenever nonprismatic beams are considered in the papers, then a specific form of the functions describing the system's variable parameters, e.g. linear functions, parabolic functions and exponential functions (as in Ref. [8]) or power polynomials (as in Ref. [9]), is used. In a general case, the solutions are not known.

The present paper deals with the problem of linear vibration of the Timoshenko beam with variable strength and geometric parameters, resting on a two-parameter nonhomogenous elastic foundation. It is assumed that the functions which describe the beam's variable parameters such as: flexural rigidity, density, variable foundation parameters and loads can be described by any functions, provided that they can be expanded into convergent series relative to Chebyshev polynomials of the first kind. Using the theorems and relations for such polynomials, presented in monograph [16], the solution is obtained in the form of Chebyshev series. This method is applied to solve three problems: the prismatic beam free vibration problem, the problem of stability of nonprismatic beams loaded with a nonpotential load and the problem of stability of a frame system. Examples, selected so as to make possible comparisons of the results obtained here with results obtained by other methods, are provided to confirm the correctness and accuracy of the proposed method. In order to test the effectiveness of the proposed method the eigenvalue problem for the prismatic beam was also solved by other approximate methods, i.e. the finite element method (a method with independent approximation of the angular displacement resulting from nondilatational deformability, put forward in a paper by Langer and Bryja [18]) and approximation methods in which a solution in the form of conventional power series or conventional (trigonometric) Fourier series is sought. The obtained solutions were compared with exact analytical solutions [3]. The results of the comparisons show that the proposed method's errors are much smaller (relative to the exact solutions) than those of the other methods tested in the example. In the case of the second problem, the solutions obtained for the Timoshenko beam are compared with the ones for Euler beams and examples are solved to show the influence of a two-parameter foundation on critical load values. The third case shows that the method can be applied to more complex beam systems, i.e. frame systems.

2. Formulation of the problem

A rectilinear nonprismatic Timoshenko beam with length $2a$, resting on a two-parameter elastic foundation, is considered. The beam is subjected to normal dynamic force loads $p(X, t)$, moment loads $o(X, t)$, tangent load

$r(X, t)$ and static load $F(X)$ which is a tangential load for $\eta = 1$ and an axial load for $\eta = 0$. $F(X) > 0$ is assumed for the compressive force. It is also assumed that all the beam's geometrical and material parameters are symmetric relative to axis X. If the beam is transversely nonhomogenous relative to axis X, then the nonhomogeneity is symmetric at the most.

The linear transverse and longitudinal vibrations of the beam are described by the following system of differential equations [8]:

$$
\begin{aligned}
&\frac{\partial Q}{\partial X} - \eta F(X)\frac{\partial^2 W}{\partial X^2} - \rho_V(X)\frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial X}\left(C(X)\frac{\partial W}{\partial X}\right) - K(X)W + p(X,t) = 0,\\
&Q - \frac{\partial M}{\partial X} + (1-\eta)F(X)\frac{\partial W}{\partial X} - I(X)\frac{\partial^2 \Phi}{\partial t^2} + o(X,t) = 0,\\
&\frac{\partial}{\partial X}\left(EA(X)\frac{\partial U}{\partial X}\right) - H(X)U - \rho_H(X)\frac{\partial^2 U}{\partial t^2} + r(X,t) = 0,
\end{aligned}
\tag{1}
$$

where W is the displacement perpendicular to the beam's axis, U the displacement tangent to the axis, Φ the angle of rotation of the cross section, M the bending moment and Q the shearing force. Functions $\rho_{V,H}(X) = \rho_B(X) + \rho_F^{V,H}(X)$ are the sum of the beam's masses and the so-called interacting foundation masses, per unit length. $I(X) = I_B(X)$ is the beam cross section's mass moment of inertia (per unit beam length). $K(X)$, $C(X)$, $H(X)$ are functions which characterize the reactions of the elastic foundation. Beam segment dX is shown in Fig. 1.

The influence of the second-order effects, due to the action of time-variable axial force $S = EA(X)\partial U/\partial X$, on the transverse vibration of the beam is neglected in Eq. (1). Otherwise, a conjugated system of nonlinear equations would be obtained. Conjugations in system (1) also appear when the parameters are not symmetric relative to axis X.

If one uses the formulas for bending moments, shearing forces and axial forces

$$
\begin{gathered}
M(X,t) = -EJ(X)\frac{\partial \Phi}{\partial X}, \quad Q(X,t) = \kappa GA(X)\beta + F(X)\left(\eta\frac{\partial W}{\partial X} - \Phi\right), \quad \beta = \frac{\partial W}{\partial X} - \Phi,\\
S(X,t) = EA(X)\frac{\partial U}{\partial X},
\end{gathered}
\tag{2}
$$

where $EJ(X)$ is the beam's flexural rigidity, $EA(X)$ its axial rigidity, $\kappa GA(X)$ its shear rigidity multiplied by cross-sectional shape factor κ and β is the angle of nondilatational strain, system of Eqs. (1) will be expressed

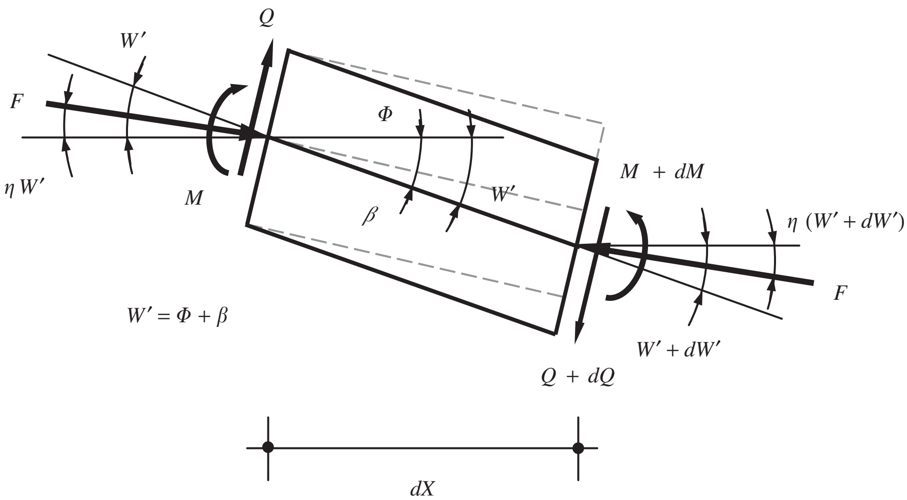

Fig. 1. Beam segment dX under axial ($\eta = 0$) or tangential ($\eta = 1$) loads.

by this formula:

$$
\begin{cases}
(\kappa GA + C)\dfrac{\partial^2 W}{\partial X^2} + \left(\dfrac{\partial \kappa GA}{\partial X} + \dfrac{\partial(\eta F)}{\partial X} + \dfrac{\partial C}{\partial X}\right)\dfrac{\partial W}{\partial X} - KW \\
\quad -(\kappa GA + F)\dfrac{\partial \Phi}{\partial X} - \left(\dfrac{\partial \kappa GA}{\partial X} + \dfrac{\partial F}{\partial X}\right)\Phi - \rho_V \dfrac{\partial^2 W}{\partial t^2} + p = 0, \\
(\kappa GA + F)\dfrac{\partial W}{\partial X} + EJ\dfrac{\partial^2 \Phi}{\partial X^2} + \dfrac{\partial EJ}{\partial X}\dfrac{\partial \Phi}{\partial X} - (\kappa GA + F)\Phi - I\dfrac{\partial^2 \Phi}{\partial t^2} + o = 0, \\
EA\frac{\partial^2 U}{\partial X^2} + \dfrac{\partial EA}{\partial X}\dfrac{\partial U}{\partial X} - HU - \rho_H \dfrac{\partial^2 U}{\partial t^2} + r = 0.
\end{cases}
\tag{3}
$$

If nondimensional variables $x = X/a$, $w(x) = W(X)/a$, $u(x) = U(X)/a$, $\phi(x) = \Phi(X)$, are introduced, system of Eqs. (3) assumes this form

$$
\begin{cases}
\left(\overline{\kappa GA} + n\overline{C}\right)\dfrac{\partial^2 w}{\partial x^2} + \left(\dfrac{\partial \overline{\kappa GA}}{\partial x} + n\left(\dfrac{\partial(\bar{\eta}\overline{F})}{\partial x} + \dfrac{\partial \overline{C}}{\partial x}\right)\right)\dfrac{\partial w}{\partial x} - n\overline{K}w \\
\quad -\left(\overline{\kappa GA} + n\overline{F}\right)\dfrac{\partial \phi}{\partial x} - \left(\dfrac{\partial \overline{\kappa GA}}{\partial x} + n\dfrac{\partial \overline{F}}{\partial x}\right)\phi - g\bar{\rho}_V \dfrac{\partial^2 w}{\partial t^2} + n\bar{p} = 0, \\
\left(\overline{\kappa GA} + n\overline{F}\right)\dfrac{\partial w}{\partial x} + \overline{EJ}\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial \overline{EJ}}{\partial x}\dfrac{\partial \phi}{\partial x} - \left(\overline{\kappa GA} + n\overline{F}\right)\phi - g\overline{I}\dfrac{\partial^2 \phi}{\partial t^2} + n\bar{o} = 0, \\
\quad d\left(\overline{EA}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial \overline{EA}}{\partial x}\dfrac{\partial u}{\partial x}\right) - n\overline{H}u - g\bar{\rho}_H \dfrac{\partial^2 u}{\partial t^2} + n\bar{r} = 0,
\end{cases}
\tag{4}
$$

and the cross-sectional forces (2) are expressed by

$$
\begin{aligned}
&m(x,t) = \frac{M(X,t)}{EJ_0} = -\overline{EJ}\frac{\partial \phi}{\partial x}, \quad q(x,t) = \frac{Q(ax)a^2}{EJ_0} = \overline{\kappa GA}\left(\frac{\partial w}{\partial x} - \phi\right) + n\overline{F}\left(\bar{\eta}\frac{\partial w}{\partial x} - \phi\right), \\
&s(x,t) = \frac{S(ax)}{EA_0} = \overline{EA}\frac{\partial u}{\partial x},
\end{aligned}
\tag{5}
$$

where

$$
\begin{gathered}
EJ(X) = EJ_0\overline{EJ}(x), \quad \kappa GA(X) = \frac{EJ_0}{a^2}\overline{\kappa GA}(x), \quad EA(X) = EA_0\overline{EA}(x), \quad F(X) = P_0\overline{F}(x), \\
\eta(X) = \bar{\eta}(x), \quad \rho_{V,H} = \rho_0\bar{\rho}_{V,H}, \quad I(X) = \rho_0 a^2\overline{I}(x), \quad n = \frac{a^2P_0}{EJ_0}, \quad g = \frac{a^4\rho_0}{EJ_0}, \quad d = \frac{a^2EA_0}{EJ_0}, \\
p(X,t) = \frac{P_0}{a}\bar{p}(x,t), \quad o(X,t) = P_0\bar{o}(x,t), \quad r(X,t) = \frac{P_0}{a}\bar{r}(x,t), \\
C(X) = P_0\overline{C}(x), \quad K(X) = \frac{P_0}{a^2}\overline{K}(x), \quad H(X) = \frac{P_0}{a^2}\overline{H}(x)
\end{gathered}
\tag{6}
$$

and EJ_0, EA_0, ρ_0, P_0 are reference quantities. In order to simplify the notation, EJ, κGA, EA, F, C, K, H, I, $\rho_{V,H}$, η, p, o, r are consistently used for $\overline{EJ}, \overline{\kappa GA}, \overline{EA}, \overline{F}, \overline{C}, \overline{K}, \overline{H}, \overline{I}, \bar{\rho}_{V,H}, \bar{\eta}, \bar{p}, \bar{o}, \bar{r}$. Symbols ρ_V and ρ_H will not be distinguished and will be represented by one symbol ρ bearing in mind that $\rho = \rho_V$ in the equations for transverse vibration and $\rho = \rho_H$ in the equations for longitudinal vibration.

3. Solution of the problem

A solution of system of Eqs. (4) is sought in the form of Chebyshev series of first kind:

$$
w(x,t) = {\sum_{l=0}^{\infty}}' a_l[w]\mathrm{T}_l(x) = {\sum_{l=0}^{\infty}}' w_l\mathrm{T}_l(x),
$$

$$\phi(x,t) = \sum_{l=0}^{\infty}{}' a_l[\phi]\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' \phi_l \mathrm{T}_l(x),$$
$$u(x,t) = \sum_{l=0}^{\infty}{}' a_l[u]\mathrm{T}_l(x) = \sum_{l=0}^{\infty}{}' u_l \mathrm{T}_l(x), \tag{7}$$

where

$$\sum_{l=0}^{\infty}{}' a_l[f] = \frac{1}{2}a_0[f] + a_1[f] + a_2[f] + \ldots, \tag{8}$$

$\mathrm{T}_l\,(x)$ is the l-th Chebyshev polynomial of 1st kind and $a_l\,[w]$, $a_l\,[u]$, $a_l\,[\phi]$ are unknown coefficients of expansion of displacement function w, u and cross section rotation function ϕ into Chebyshev series, further denoted by, respectively w_l, u_l and ϕ_l.

The theorem on the method of solving ordinary differential equations, presented in Appendix A, was applied to solve system (4).

Using the denotations from the Appendix A theorem one can express system of Eqs. (4) in the following matrix form:

$$\hat{\mathbf{P}}_0(x)\mathbf{f}''(x,t) + \hat{\mathbf{P}}_1(x)\mathbf{f}'(x,t) + \hat{\mathbf{P}}_2(x)\mathbf{f}(x,t) = \hat{\mathbf{P}}(x,t), \tag{9}$$

where

$$\hat{\mathbf{P}}_0(x) = \begin{bmatrix} \kappa GA + nC & 0 & 0 \\ 0 & EJ & 0 \\ 0 & 0 & d\,EA \end{bmatrix},$$
$$\hat{\mathbf{P}}_1(x) = \begin{bmatrix} \dfrac{\partial \kappa GA}{\partial x} + n\left(\dfrac{\partial(\eta F)}{\partial x} + \dfrac{\partial C}{\partial x}\right) & -(\kappa GA + nF) & 0 \\ \kappa GA + nF & \dfrac{\partial EJ}{\partial x} & 0 \\ 0 & 0 & d\dfrac{\partial EA}{\partial x} \end{bmatrix},$$
$$\hat{\mathbf{P}}_2(x) = -\begin{bmatrix} nK & \dfrac{\partial \kappa GA}{\partial x} + n\dfrac{\partial F}{\partial x} & 0 \\ 0 & \kappa GA + nF & 0 \\ 0 & 0 & nH \end{bmatrix},$$
$$\hat{\mathbf{P}}(x,t) = \begin{bmatrix} g\rho\ddot{w} - np \\ gI\ddot{\phi} - no \\ g\rho\ddot{u} - nr \end{bmatrix} = \begin{bmatrix} g\rho & 0 & 0 \\ 0 & gI & 0 \\ 0 & 0 & g\rho \end{bmatrix}\begin{bmatrix} \ddot{w} \\ \ddot{\phi} \\ \ddot{u} \end{bmatrix} - \begin{bmatrix} np \\ no \\ nr \end{bmatrix} = \mathbf{B}\ddot{\mathbf{f}}(x,t) - \hat{\mathbf{P}}_{\text{out}}(x,t) \tag{10}$$

and

$$\mathbf{f} = \begin{bmatrix} w \\ \phi \\ u \end{bmatrix}, \quad \ddot{\mathbf{f}} = \begin{bmatrix} \ddot{w} \\ \ddot{\phi} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} \partial^2 w/\partial t^2 \\ \partial^2 \phi/\partial t^2 \\ \partial^2 u/\partial t^2 \end{bmatrix}. \tag{11}$$

Functions $\mathbf{Q_m}(x)$ defined by formula (A.8) assume this form

$$\mathbf{Q}_0(x) = \begin{bmatrix} \kappa GA + nC & 0 & 0 \\ 0 & EJ & 0 \\ 0 & 0 & dEA \end{bmatrix},$$

$$\mathbf{Q}_1(x) = \begin{bmatrix} -\frac{\partial \kappa GA}{\partial x} + n\left(\frac{\partial(\eta F)}{\partial x} - \frac{\partial C}{\partial x}\right) & -(\kappa GA + nF) & 0 \\ \kappa GA + nF & -\frac{\partial EJ}{\partial x} & 0 \\ 0 & 0 & -d\frac{\partial EA}{\partial x} \end{bmatrix},$$

$$\mathbf{Q}_2(x) = -\begin{bmatrix} n\frac{\partial^2(\eta F)}{\partial x^2} + nK & 0 & 0 \\ \frac{\partial \kappa GA}{\partial x} + n\frac{\partial F}{\partial x} & \kappa GA + nF & 0 \\ 0 & 0 & nH \end{bmatrix}. \tag{12}$$

After determining coefficients $b_{nmj}(k)$ in Eq. (A.3) and applying the following relation for the coefficients of expansion of the product of two functions (see Ref. [16 p. 128, (33)]):

$$a_l[f(x)\cdot g(x)] = \frac{1}{2}\sum_{m=0}^{\infty}{}' a_m[f](a_{l-m}[g] + a_{l+m}[g]), \tag{13}$$

one obtains the following relation, equivalent to formula (A.3):

$$\begin{aligned} &\sum_{l=0}^{\infty}{}' \{2(k^2-1)k(a_{k-l}[\mathbf{Q}_0] + a_{k+l}[\mathbf{Q}_0]) \\ &\quad + (k^2-1)(a_{k-l-1}[\mathbf{Q}_1] + a_{k+l-1}[\mathbf{Q}_1] - a_{k-l+1}[\mathbf{Q}_1] - a_{k+l+1}[\mathbf{Q}_1]) \\ &\quad + \frac{1}{2}((k+1)(a_{k-l-2}[\mathbf{Q}_2] + a_{k+l-2}[\mathbf{Q}_2]) - 2k(a_{k-l}[\mathbf{Q}_2] + a_{k+l}[\mathbf{Q}_2]) \\ &\quad + (k-1)(a_{k-l+2}[\mathbf{Q}_2] + a_{k+l+2}[\mathbf{Q}_2]))\}a_l[\mathbf{f}] \\ &= \frac{1}{2}\sum_{l=0}^{\infty}{}' \{(k+1)(a_{k-l-2}[\mathbf{B}] + a_{k+l-2}[\mathbf{B}]) - 2k(a_{k-l}[\mathbf{B}] + a_{k+l}[\mathbf{B}]) + (k-1)(a_{k-l+2}[\mathbf{B}] + a_{k+l+2}[\mathbf{B}])\}a_l[\ddot{\mathbf{f}}] \\ &\quad - n\Big((k+1)a_{k-2}[\hat{\mathbf{P}}_{\text{out}}] - 2ka_k[\hat{\mathbf{P}}_{\text{out}}] + (k-1)a_{k+2}[\hat{\mathbf{P}}_{\text{out}}]\Big) \end{aligned} \tag{14}$$

true for each integer k.

After introducing the following denotations for the coefficients of the Chebyshev expansions of the functions occurring in formula (14):

$$\begin{aligned} &a_l[EJ] = e_l, \quad a_l[\kappa GA] = a_l, \quad a_l[EA] = d_l, \quad a_l[F] = f_l, \quad a_l[C] = c_l, \quad a_l[K] = k_l, \quad a_l[H] = h_l, \\ &a_l[\rho] = g_l, \quad a_l[I] = i_l, \quad a_l[p] = p_l, \quad a_l[o] = o_l, \quad a_l[r] = r_l, \end{aligned} \tag{15}$$

assuming that $\eta(x) = \eta = \text{const}$ and performing transformations using this relation

$$a_l = \frac{1}{2l}\left(a'_{l-1} - a'_{l+1}\right), \quad l \neq 0, \tag{16}$$

where $a_l = a_l[f]$ and $a'_l = a_l[\partial f/\partial x]$, one obtains the following infinite system of ordinary differential equations:

$$\sum_{l=0}^{\infty}{}' \begin{bmatrix} k_{11}(k,l) & k_{12}(k,l) & 0 \\ k_{21}(k,l) & k_{22}(k,l) & 0 \\ 0 & 0 & k_{33}(k,l) \end{bmatrix} \begin{bmatrix} w_l \\ \phi_l \\ u_l \end{bmatrix} = \begin{bmatrix} P_1(k) \\ P_2(k) \\ P_3(k) \end{bmatrix} + \sum_{l=0}^{\infty}{}' \begin{bmatrix} m_{11}(k,l) & 0 & 0 \\ 0 & m_{22}(k,l) & 0 \\ 0 & 0 & m_{33}(k,l) \end{bmatrix} \begin{bmatrix} \ddot{w}_l \\ \ddot{\phi}_l \\ \ddot{u}_l \end{bmatrix},$$
$$k = 0, 1, 2, 3, \ldots, \tag{17}$$

where

$$\begin{aligned} k_{11}(k,l) = {} & 2(k^2-1)\,l\,[n(c_{k-l}-c_{k+l}) + (a_{k-l}-a_{k+l})] \\ & - \frac{1}{2}n[(k+1)(k_{k-l-2}-k_{k+l-2}) - 2k(k_{k-l}-k_{k+l}) + (k-1)(k_{k-l+2}-k_{k+l+2})] \\ & + 2n\eta\left[(k+1)(k-l)\,l f_{k-l} - (k-1)(k+l)\,l f_{k-l}\right] \\ & + \begin{cases} 0 & \text{for } l = 0, 1, \\ 4n\eta l \sum_{j=1}^{l-1}(k-l+2j) f_{k-l+2j} & \text{for } l \geqslant 2, \end{cases} \end{aligned}$$

$$k_{12}(k,l) = -(k^2-1)\left[(a_{k-l-1} + a_{k+l-1} - a_{k-l+1} - a_{k+l+1}) + n\left(f_{k-l-1} + f_{k+l-1} - f_{k-l+1} - f_{k+l+1}\right)\right], \tag{18}$$

$$\begin{aligned} k_{21}(k,l) = {} & (k+1)l(a_{k-l-1}-a_{k+l-1}) - (k-1)l(a_{k-l+1}-a_{k+l+1}) \\ & + n\left[(k+1)\,l\left(f_{k-l-1} - f_{k+l-1}\right) - (k-1)\,l\left(f_{k-l+1} - f_{k+l+1}\right)\right], \end{aligned}$$

$$\begin{aligned} k_{22}(k,l) = {} & 2(k^2-1)l(e_{k-l}-e_{k+l}) \\ & - \frac{1}{2}\Big[(k+1)(a_{k-l-2}+a_{k+l-2}) - 2k(a_{k-l}+a_{k+l}) + (k-1)(a_{k-l+2}+a_{k+l+2}) \\ & + n\left((k+1)\left(f_{k-l-2}+f_{k+l-2}\right) - 2k\left(f_{k-l}+f_{k+l}\right) + (k-1)\left(f_{k-l+2}+f_{k+l+2}\right)\right)\Big], \end{aligned}$$

$$\begin{aligned} k_{33}(k,l) = {} & 2d(k^2-1)l(d_{k-l}-d_{k+l}) \\ & - \frac{1}{2}n[(k+1)(h_{k-l-2}-h_{k+l-2}) - 2k(h_{k-l}-h_{k+l}) + (k-1)(h_{k-l+2}-h_{k+l+2})], \end{aligned}$$

$$\begin{aligned} P_1(k) &= -n\left[(k+1)p_{k-2} - 2kp_k + (k-1)p_{k+2}\right], \\ P_2(k) &= -n[(k+1)o_{k-2} - 2ko_k + (k-1)o_{k+2}], \\ P_3(k) &= -n[(k+1)r_{k-2} - 2kr_k + (k-1)r_{k+2}], \end{aligned} \tag{19}$$

$$\begin{aligned} m_{11}(k,l) &= \frac{1}{2}g\sum_{l=0}^{\infty}{}' \left[(k+1)\left(g_{k-l-2}+g_{k+l-2}\right) - 2k\left(g_{k-l}+g_{k+l}\right) + (k-1)\left(g_{k-l+2}+g_{k+l+2}\right)\right], \\ m_{22}(k,l) &= \frac{1}{2}g\sum_{l=0}^{\infty}{}' [(k+1)(i_{k-l-2}+i_{k+l-2}) - 2k(i_{k-l}+i_{k+l}) + (k-1)(i_{k-l+2}+i_{k+l+2})], \\ m_{33}(k,l) &= \frac{1}{2}g\sum_{l=0}^{\infty}{}' \left[(k+1)\left(g_{k-l-2}+g_{k+l-2}\right) - 2k\left(g_{k-l}+g_{k+l}\right) + (k-1)\left(g_{k-l+2}+g_{k+l+2}\right)\right]. \end{aligned} \tag{20}$$

The transformation of Eqs. (14) is described in more detail for a fourth-order differential equation in earlier works by the author [1,2].

In infinite system of Eqs. (17) the first two groups of (six) equations are satisfied identity-wise for $k = 0,1$ (the number of groups of equations satisfied identity-wise is equal to the order of the differential equation). This equations are replaced by equations for boundary conditions. In the case of the Timoshenko beam, the boundary conditions for the basic modes of support are as follows:

- pin supported end

$$w = 0, \quad m = -EJ\frac{\partial \phi}{\partial x} = 0, \quad u = 0 \quad \text{or} \quad s = EA\frac{\partial u}{\partial x} = 0. \tag{21}$$

- clamped end

$$w = 0, \quad \phi = 0, \quad u = 0. \tag{22}$$

- Free end

$$\begin{aligned} m &= -EJ\frac{\partial \phi}{\partial x} = 0, \\ q &= \kappa GA\left(\frac{\partial w}{\partial x} - \phi\right) + nF\left(\eta\frac{\partial w}{\partial x} - \phi\right) = 0, \\ s &= EA\frac{\partial u}{\partial x} = 0. \end{aligned} \tag{23}$$

The following formulas stating the values of the Chebyshev polynomials and their first derivatives at points ∓ 1 (coordinates of the beam's ends):

$$\mathrm{T}_n(1) = 1, \quad \mathrm{T}_n(-1) = (-1)^n, \quad \mathrm{T}'_n(1) = n^2, \quad \mathrm{T}'_n(-1) = -(-1)^n n^2, \tag{24}$$

are helpful when introducing the equations for boundary conditions. Having expanded the functions in definition of internal forces (5) into Chebyshev series, determined the respective derivatives and employed identity (24), one gets the following expression for kinetic boundary conditions:

$$m(-1,t) = EJ_-\sum_{l=0}^{\infty}{}' (-1)^l l^2 \phi_l, \quad m(+1,t) = -EJ_+\sum_{l=0}^{\infty}{}' l^2 \phi_l, \tag{25}$$

$$\begin{aligned} q(-1,t) &= -\sum_{l=0}^{\infty}{}' (\kappa GA + n\eta f)_-(-1)^l l^2 w_l - \sum_{l=0}^{\infty}{}' (\kappa GA + nf)_-(-1)^l \phi_l, \\ q(+1,t) &= \sum_{l=0}^{\infty}{}' (\kappa GA + n\eta f)_+ l^2 w_l - \sum_{l=0}^{\infty}{}' (\kappa GA + nf)_+ \phi_l, \end{aligned} \tag{26}$$

$$s(-1,t) = EA_-\sum_{l=0}^{\infty}{}' (-1)^l l^2 u_l, \quad s(+1,t) = EA_+\sum_{l=0}^{\infty}{}' l^2 u_l, \tag{27}$$

where

$$\begin{aligned} EJ_- &= EJ(-1) = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l e_l, \quad EJ_+ = EJ(+1) = \sum_{l=0}^{\infty}{}' e_l \mathrm{T}_l(1) = \sum_{l=0}^{\infty}{}' e_l, \\ EA_- &= EA(-1) = \sum_{l=0}^{\infty}{}' d_l \mathrm{T}_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l d_l, \quad EA_+ = EA(+1) = \sum_{l=0}^{\infty}{}' d_l \mathrm{T}_l(1) = \sum_{l=0}^{\infty}{}' d_l, \end{aligned}$$

$$(\kappa GA + nsf)_- = \kappa GA(-1) + nsf(-1) = \sum_{l=0}^{\infty}{}' (a_l + nsf_l)\mathrm{T}_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l (a_l + nsf_l),$$

$$(\kappa GA + nsf)_+ = \kappa GA(+1) + nsf(+1) = \sum_{l=0}^{\infty}{}' (a_l + nsf_l)\mathrm{T}_l(+1) = \sum_{l=0}^{\infty}{}' (a_l + nsf_l), \quad s = 1 \vee s = \eta. \tag{28}$$

The following formulas

$$w(-1,t) = \sum_{l=0}^{\infty}{}' (-1)^l w_l, \quad w(+1,t) = \sum_{l=0}^{\infty}{}' w_l,$$

$$\phi(-1,t) = \sum_{l=0}^{\infty}{}' (-1)^l \phi_l, \quad \phi(+1,t) = \sum_{l=0}^{\infty}{}' \phi_l,$$

$$u(-1,t) = \sum_{l=0}^{\infty}{}' (-1)^l u_l, \quad u(+1,t) = \sum_{l=0}^{\infty}{}' u_l \tag{29}$$

obtained from expressions (7) after relation (24) is taken into account, are used to determine kinematic boundary conditions.

By replacing the first four equations of infinite system of Eqs. (17) with the equations for boundary conditions one gets an infinite system of ordinary differential equations which allow one to solve the problem. In a special case of harmonic vibration, the infinite system of differential equations reduces itself to an infinite system of algebraic equations.

4. Numerical examples

In order to illustrate the method and verify its effectiveness, three numerical examples were solved. In the examples, the solutions obtained by the proposed method are compared with those obtained by other methods (including exact analytical solutions) and the solutions for the Timoshenko beam are compared with those for the Euler model. In the first two examples only the transverse vibrations of the beams were analysed while in the third example the axial deformability of the beams was taken into account.

4.1. Example 1

The method was applied to solve the prismatic beam natural vibration problem. Two types of beams: simple supported and clamped–free were considered. The beams' parameters were: modulus of elasticity $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$, modulus of rigidity $G = 3E/8$ beam length $2a = 0.4\,\mathrm{m}$, the cross section in the form of a rectangle with height $h = 0.08\,\mathrm{m}$ and width $b = 0.02\,\mathrm{m}$, cross-sectional shape factor $\kappa = 2/3$ and beam density $\rho_{BV} = 7850\,\mathrm{kg/m^3}$.

The results were compared with the analytical solutions obtained by Huang [3]. Since some editorial errors (among others, incorrect signs in some expressions) and an incorrectly derived (a group of solutions omitted) formula for the free vibration of the simple supported beam were found in Ref. [3], amended versions of the formulas were used for the comparisons. In the case of the free vibration problem, system of differential equations (17) (supplemented with equations for boundary conditions) becomes an infinite system of algebraic equations. To solve the equations, the system was limited to a finite system. Then the displacement function and the cross section rotation are given by the finite sums of Chebyshev series

$$w(x) = \sum_{l=0}^{m} w_l \mathrm{T}_l(x), \quad \phi(x) = \sum_{l=0}^{m} \phi_l \mathrm{T}_l(x). \tag{30}$$

To test the convergence of the solutions, the system was solved for an ever increasing size of the approximation base: $m = 20$, 30. The natural frequencies (for the different types of beams) determined by the proposed method and the exact values calculated on the basis of Ref. [3] are shown in Tables 1 and 3. Whereas the relative error between the exact solution [3] and the solution obtained by the proposed method is shown in

Table 1
Free vibration frequencies for supported–supported beam

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$m = 20$	6838.8336	23 190.827	43 443.493	64 939.185	86 710.888	108 431.32	111 981.29	120 647.23	130 006.37
$m = 30$	6838.8336	23 190.827	43 443.493	64 939.185	86 710.889	108 431.34	111 981.29	120 647.23	130 003.61
Exact	6838.8336	23 190.827	43 443.493	64 939.185	86 710.889	108 431.34	111 981.29	120 647.23	130 003.61

Table 2
Relative errors Err_m given by formula (31) for eigenforms of supported–supported beam

		$i = 1$	2	3	4	5	6	7	8	9
$m = 20$	W_i	1.44×10^{-15}	1.92×10^{-15}	4.59×10^{-11}	1.62×10^{-9}	9.06×10^{-7}	4.34×10^{-6}	0.00	6.17×10^{-13}	5.12×10^{-4}
	Φ_i	2.04×10^{-15}	6.21×10^{-15}	1.82×10^{-10}	2.52×10^{-9}	7.41×10^{-6}	4.87×10^{-6}	0.00	2.90×10^{-13}	7.05×10^{-3}
$m = 30$	W_i	1.87×10^{-15}	2.26×10^{-15}	7.47×10^{-15}	1.42×10^{-14}	4.64×10^{-14}	1.63×10^{-13}	0.00	2.86×10^{-13}	3.07×10^{-10}
	Φ_i	1.91×10^{-15}	1.51×10^{-15}	2.96×10^{-14}	3.98×10^{-14}	1.07×10^{-13}	1.75×10^{-13}	0.00	6.65×10^{-14}	4.06×10^{-9}

Table 3
Free vibration frequencies for clamped–free beam

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$m = 20$	2529.4927	13 279.905	31 044.791	50 825.834	71 565.046	91 994.772	11 0976.10	119 244.95	131 611.40
$m = 30$	2529.4927	13 279.905	31 044.791	50 825.834	71 565.047	91 994.824	11 0975.98	119 244.57	131 606.52
Exact	2529.4927	13 279.905	31 044.791	50 825.834	71 565.047	91 994.824	11 0975.98	119 244.57	131 606.52

Table 4
Relative errors Err_m given by formula (31) for eigenforms of clamped–free beam

		$i = 1$	2	3	4	5	6	7	8	9
$m = 20$	W_i	8.88×10^{-16}	3.33×10^{-15}	3.91×10^{-12}	1.30×10^{-9}	1.59×10^{-7}	6.68×10^{-6}	2.18×10^{-5}	1.43×10^{-4}	5.06×10^{-4}
	Φ_i	2.02×10^{-15}	5.57×10^{-15}	2.65×10^{-12}	1.66×10^{-9}	1.20×10^{-7}	2.17×10^{-6}	1.06×10^{-4}	1.64×10^{-4}	5.03×10^{-4}
$m = 30$	W_i	7.77×10^{-16}	3.33×10^{-15}	2.59×10^{-14}	1.22×10^{-13}	2.01×10^{-13}	6.01×10^{-13}	3.46×10^{-12}	4.12×10^{-11}	3.52×10^{-10}
	Φ_i	1.88×10^{-15}	5.40×10^{-15}	1.44×10^{-14}	7.63×10^{-14}	7.36×10^{-14}	2.52×10^{-14}	1.39×10^{-11}	4.00×10^{-11}	2.93×10^{-10}

Tables 2 and 4. The error is expressed by

$$\mathrm{Err}_m(f) = \frac{\sup_{-1 \leqslant x \leqslant 1} \left| f_m(x) - f(x) \right|}{\left\| f(x) \right\|}, \tag{31}$$

where $\left\| f(x) \right\| = \sup_{-1 \leqslant x \leqslant 1} \left| f(x) \right|$, $f(x)$ is an exact solution and $f_m(x)$ is a solution defined by a finite Chebyshev series with $m+1$ terms. The graphs of the eigenforms are shown in Figs. 2 and 3. The eigenforms were determined by the proposed method for approximation base size $m = 30$.

In order to test the effectiveness of the proposed method the eigenvalue problem for the prismatic beam was also solved by other approximate methods, i.e. the finite element method (with independent approximation of angular displacement due to nondilatational deformability [18]) and approximation methods in which a solution in the form of conventional power series or conventional (trigonometric) Fourier series was sought. In all the methods identical (or slightly larger in the case of FEM) approximation base dimensions ($m = 25$) were adopted. The obtained solutions were compared with the exact analytical solutions [3]. The computations were carried out for three types of beams: simple supported, clamped–free and clamped–pin supported. Because of this paper's limited length, only the results for the clamped–free beam are presented here. The exact

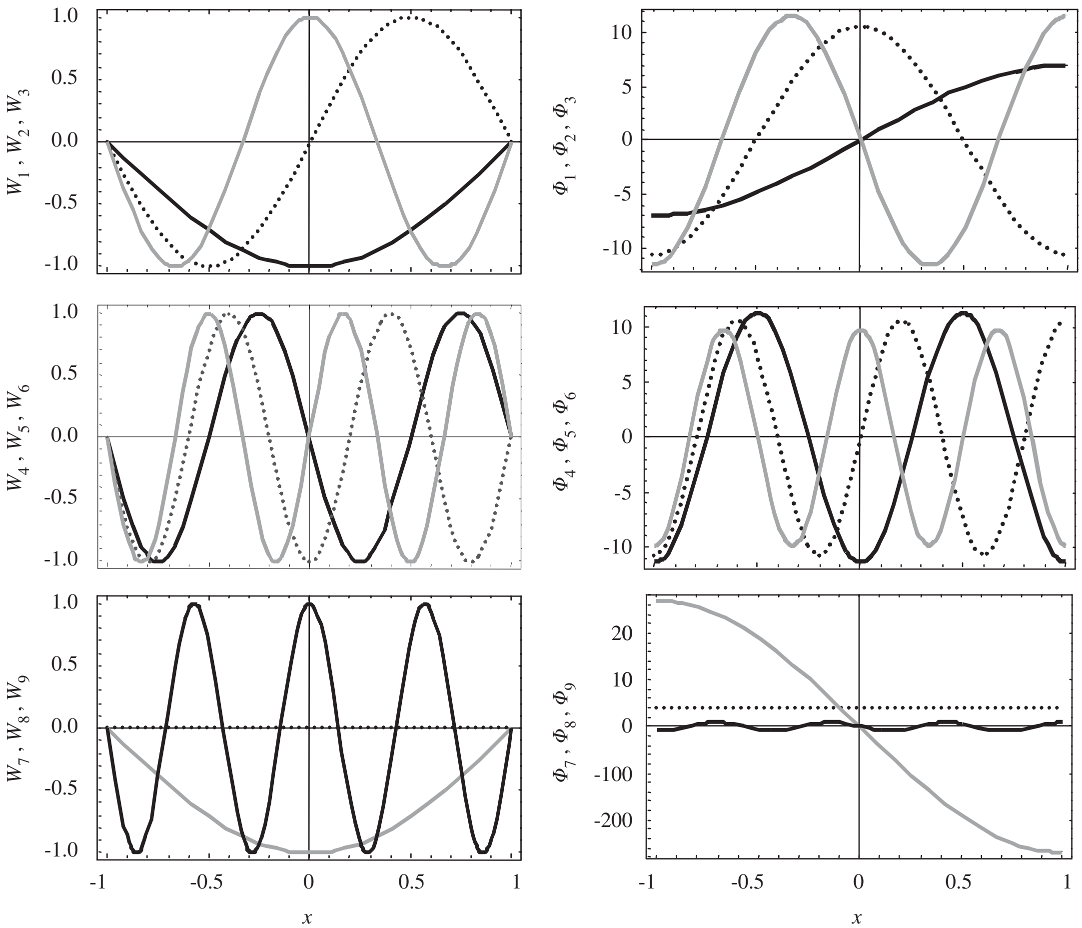

Fig. 2. Graphs of eigenforms (W_i, Φ_i) (———), (W_{i+1}, Φ_{i+1}) (.....), (W_{i+2}, Φ_{i+2}) (———), $i = 1, 4, 7$ for simple supported beams.

eigenvalues and the ones obtained by the approximate methods are shown in Table 5 where the methods are denoted by the following symbols: FEM the finite element method, CF the approximation method with solution approximation by conventional Fourier series and CP the approximation method with approximation by conventional power series. Typical graphs of relative error functions defined by this formula

$$E(f) = \frac{\left|\tilde{f}(x) - f(x)\right|}{\left\|f(x)\right\|} \tag{32}$$

for selected eigenforms (W_1, Φ_1), (W_9, Φ_9) determined by the different methods are shown in a logarithmic scale in Fig. 4. The denotations in formula (32) are identical as the ones in formula (31), except for symbol $\tilde{f}(x)$ which here stands for one of the approximate solutions.

4.2. Example 2

The problem of dynamic stability of beams under a nonpotential load was solved (Fig. 1). Two static schemes were considered: a clamped–free beam (Fig. 5a) and clamped–pin supported beam (Fig. 5b). The beams were subjected to a concentrated tangential force or a uniformly distributed tangential static load. By applying the dynamic criterion of stability loss (bifurcation or flutter), the critical values of the loads were determined. The analysis was performed for homogenous prismatic beams and beams with variable cross sections described by functions $b(x)$, $h(x)$ (Fig. 5). The following were also assumed: modulus of elasticity $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$, modulus of rigidity $G = 3E/8$, beam length $L = 2a = 0.4\,\mathrm{m}$, the cross section in the

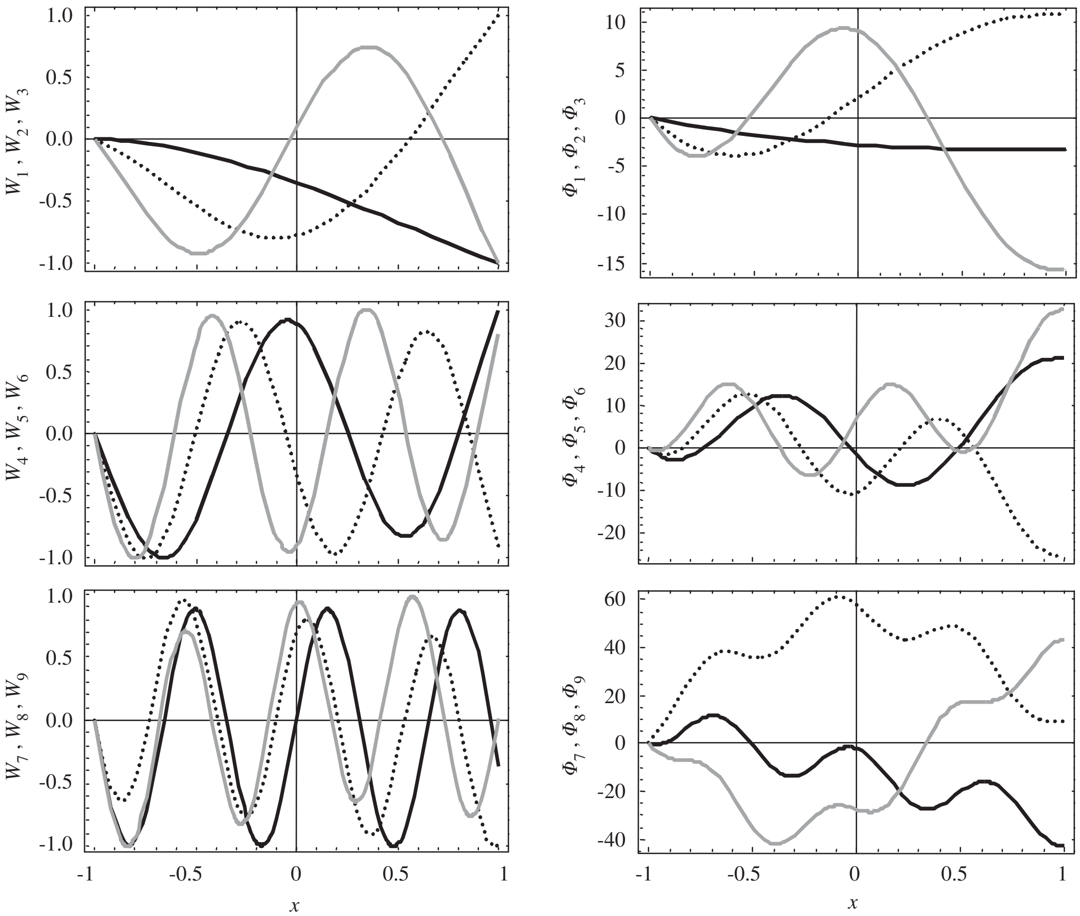

Fig. 3. Graphs of eigenforms (W_i, Φ_i) (——), (W_{i+1}, Φ_{i+1}) (.....), (W_{i+2}, Φ_{i+2}) (——), $i = 1, 4, 7$ for clamped–free beam.

Table 5
Free vibration frequencies of clamped–free beam for different solution methods

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
FEM	2529.5182	13 283.393	31 088.060	51 011.254	72 075.176	93 025.284	11 2457.31	119 997.12	133 427.40
CF	2529.3240	13 294.154	31 082.644	50 943.476	71 763.763	92 403.849	11 1681.07	119 271.29	132 393.69
CP	2529.4927	13 279.905	31 043.365	49 008.433	52 960.403– 11441.036i	52 960.403 + 11441.036i	55 118.411– 31897.209i	55 118.411 + 31 897.209i	79 570.979– 12 157.165i
This paper	2529.4927	13 279.905	31 044.791	50 825.834	71 565.047	91 994.824	11 0975.98	119 244.57	131 606.52
Exact	2529.4927	13 279.905	31 044.791	50 825.834	71 565.047	91 994.824	11 0975.98	119 244.57	131 606.52

form of a rectangle with height $h(x)$ and width $b(x)$, rectangular cross section shape factor $\kappa = 2/3$ and beam density $\rho_{BV} = 7850\,\mathrm{kg/m^3}$.

The Beck column and the Leipholz column (clamped–free beams) with $b(x) = h(x) = 0.04\,\mathrm{m}$, subjected to a concentrated tangential force or a uniformly distributed tangential load were analysed. The results are shown against the solution for the Euler beam (with the same parameters as those of the Timoshenko beam) in Fig. 6. The solutions of the stability problem for the clamped–free beams with variable cross sections described by respectively functions $b(x)=h(x)=0.04\times\sqrt{\frac{15}{43}}\left(2-(x+1)^2/4\right)\,\mathrm{m}$ and $b(x) = h(x) = 0.04 \times \sqrt{\frac{3}{7}}\left(2-(x+1)/2\right)\,\mathrm{m}$ are shown in Figs. 7a and b. Multipliers $\sqrt{\frac{15}{43}}$ and $\sqrt{\frac{3}{7}}$ in formulas for $b(x)$, $h(x)$ were so chosen that all the

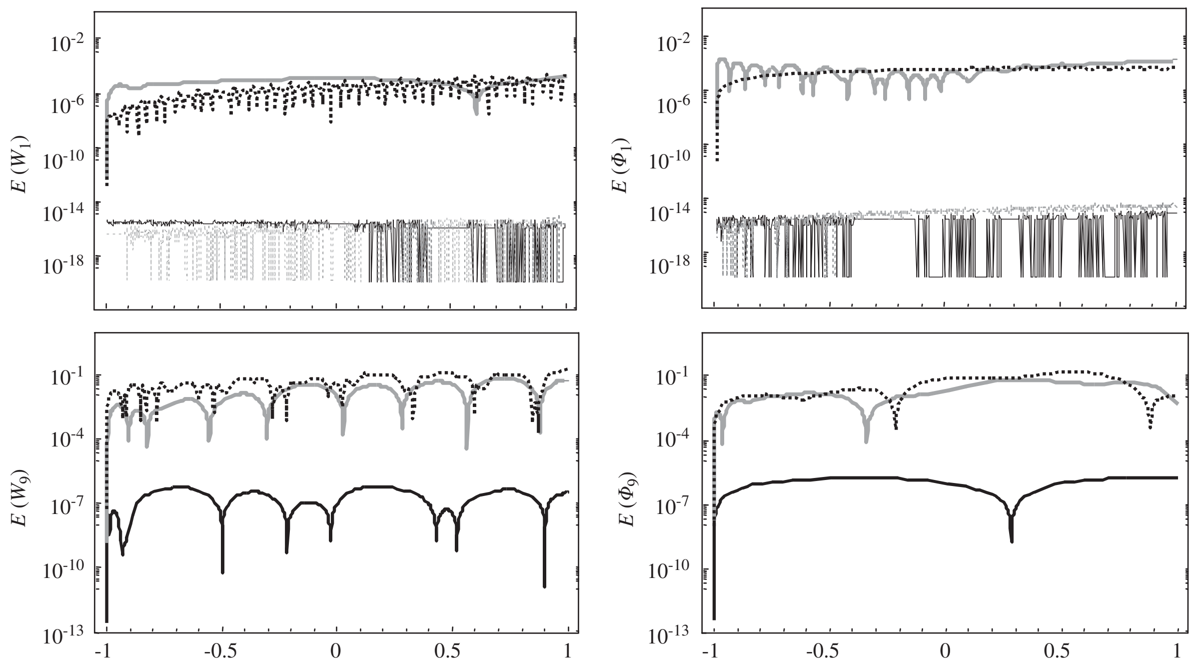

Fig. 4. Graphs of relative errors expressed by formula (32) for different methods of solving of natural vibration problem for clamped–free beam. Methods denoted as follows: proposed method (▬▬), approximation by conventional Fourier series (▬▬), FEM (.....), approximation by conventional power series (▪▪▪▪▪).

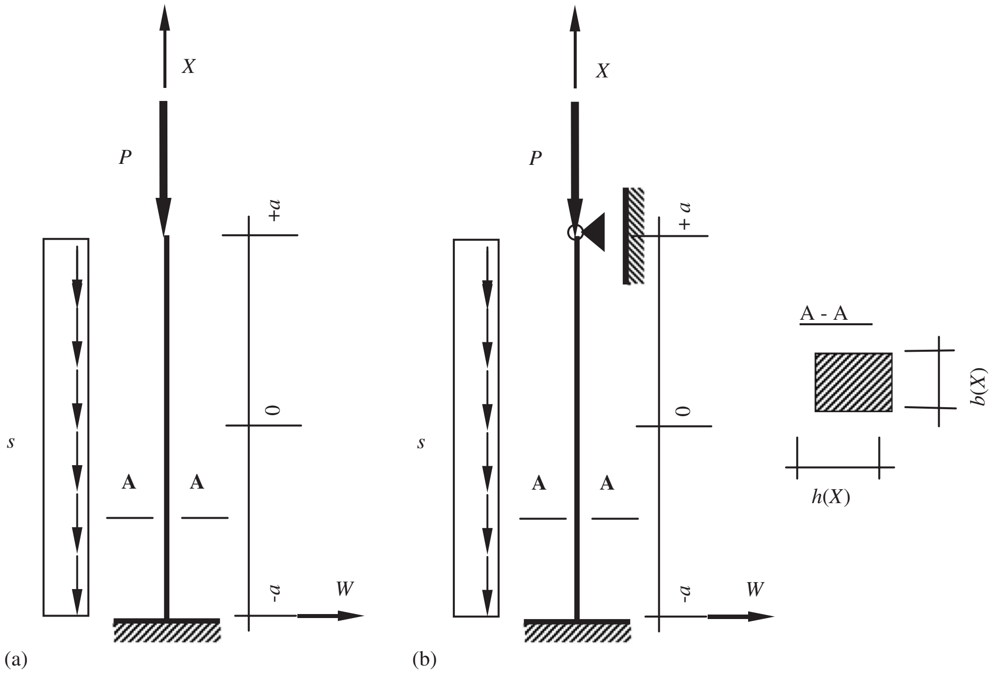

Fig. 5. Clamped–free beam (a) and clamped–pin supported beam (b) with nonpotential concentrated force P or uniformly distributed nonpotential load s.

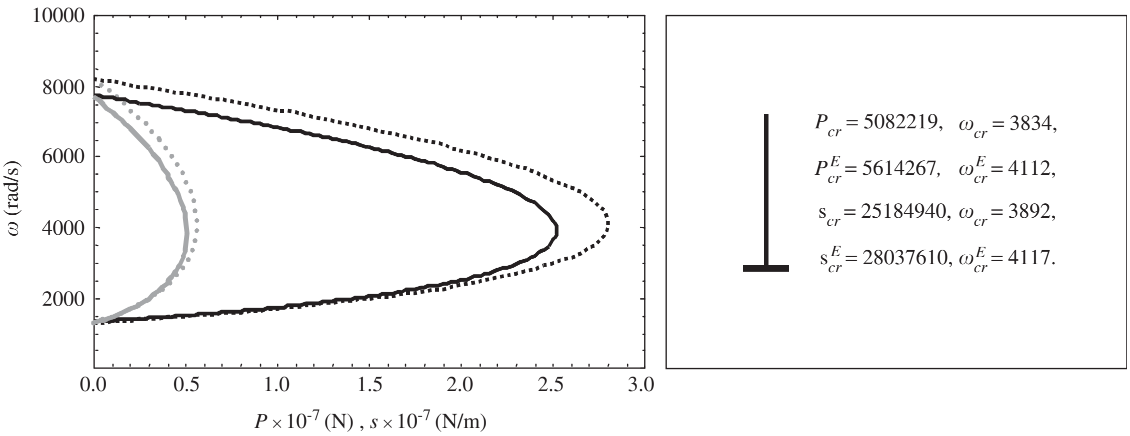

Fig. 6. Free vibration frequency versus concentrated tangential load P and distributed tangential load s for prismatic Beck and Leipholz columns with $b(x) = h(x) = 0.04$ for Timoshenko beam (T) and Euler beam (E). Symbols: load P—beam T (grey solid line), load P—beam E (grey dotted line), load s—beam T (black solid line), load s—beam E (.....).

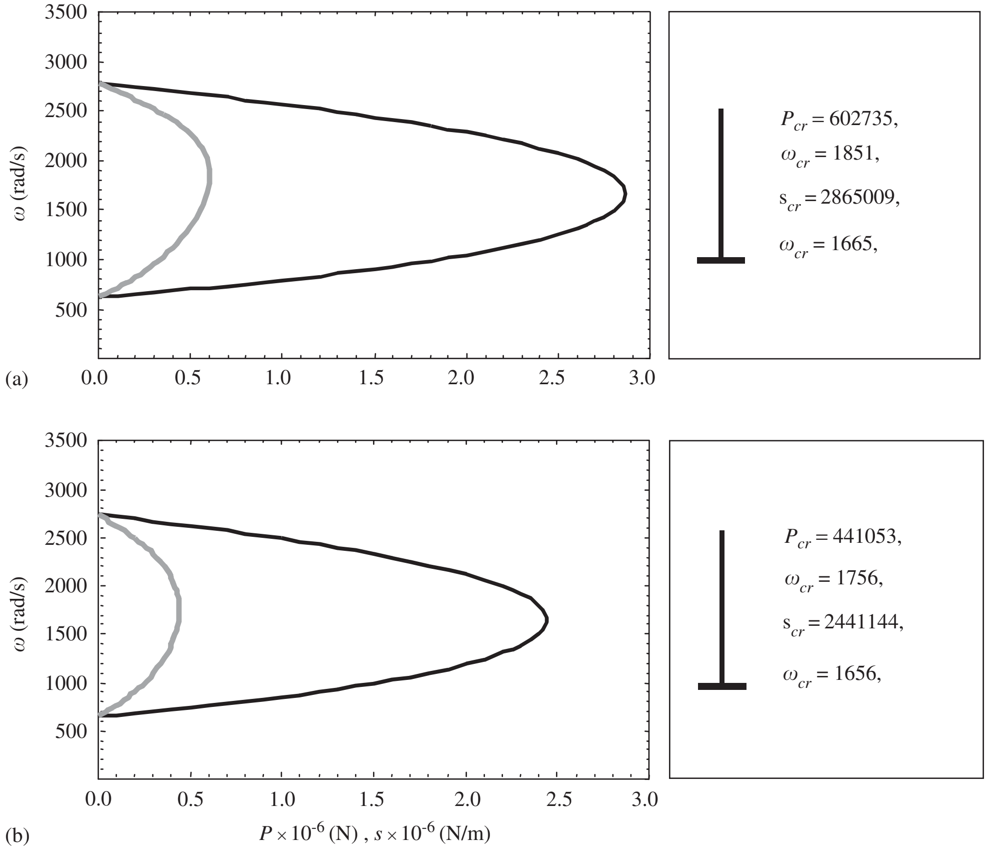

Fig. 7. Free vibration of clamped–free beam with variable cross section: (a) $b(x) = h(x) = 0.04 \times \sqrt{\frac{15}{43}\left(2 - (x+1)^2/4\right)}$; (b) $b(x) = h(x) = 0.04 \times \sqrt{\frac{3}{7}\left(2 - (x+1)/2\right)}$ versus concentrated tangential load P (grey solid line) and distributed tangential load s (black solid line).

beams had the same volume of $0.04 \times 0.04 \times 0.4\,\mathrm{m}^3$. In the considered cases, stability was lost through flutter. Similar analyses were performed for the clamped–pin supported beams for the same parameters as those of the clamped–free systems. The obtained results are shown in Figs. 8 and 9.

Fig. 10 shows the solution of the stability problem for a clamped–free prismatic beam resting on a one- (the Winkler foundation) and two-parameter elastic foundation. For comparison, the solutions which do not take into account the influence of the elastic foundation are included in Figs. 10a–c. The

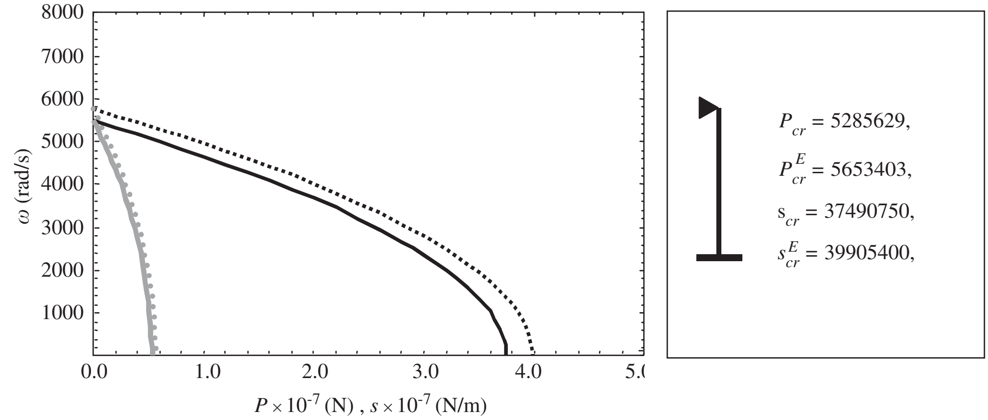

Fig. 8. Free vibration of clamped–pin supported beam versus concentrated tangential load P and distributed tangential load s for Timoshenko beam (T) and Euler beam (E) $b(x) = h(x) = 0.04$. Symbols: load P—beam T (——), load P—beam E (······), load s -beam T (——), load s—beam E (.....).

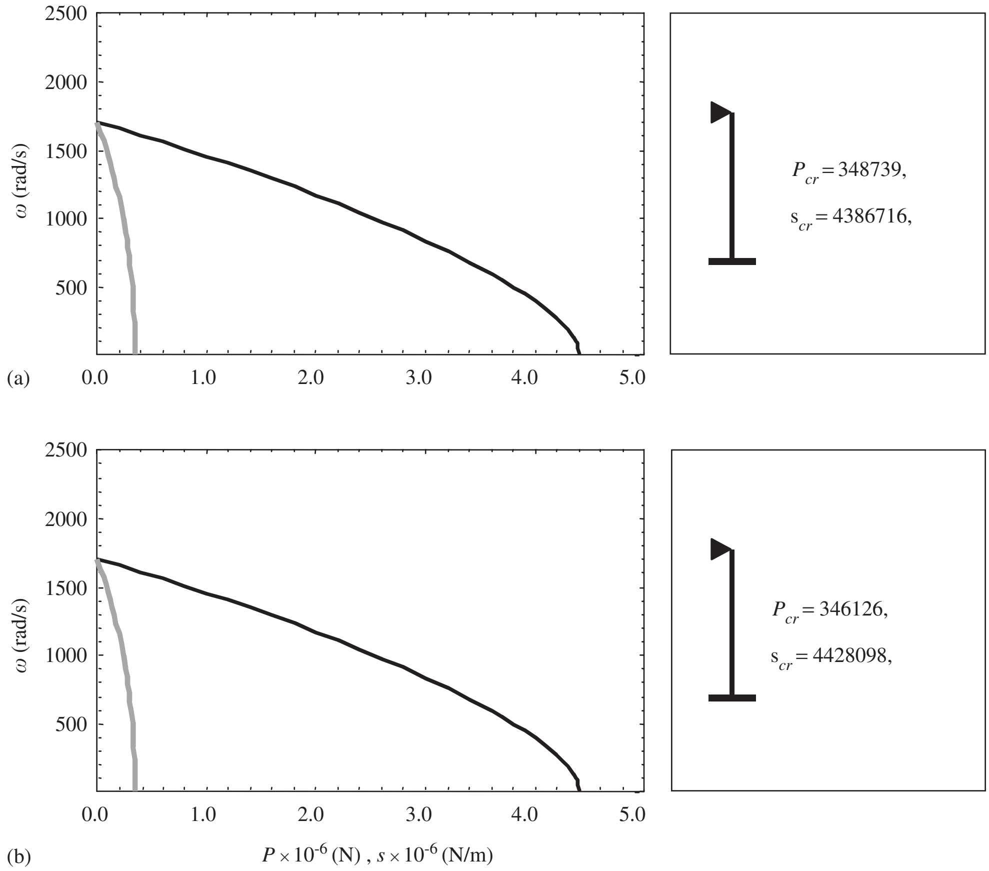

Fig. 9. Free vibration of clamped–pin supported beam with variable cross section: (a) $b(x) = h(x) = 0.04 \times \sqrt{\frac{15}{43}}\left(2-(x+1)^2/4\right)$; (b) $b(x) = h(x) = 0.04 \times \sqrt{\frac{3}{7}}\left(2-(x+1)/2\right)$ versus concentrated tangential load P (——) and distributed tangential load s (——).

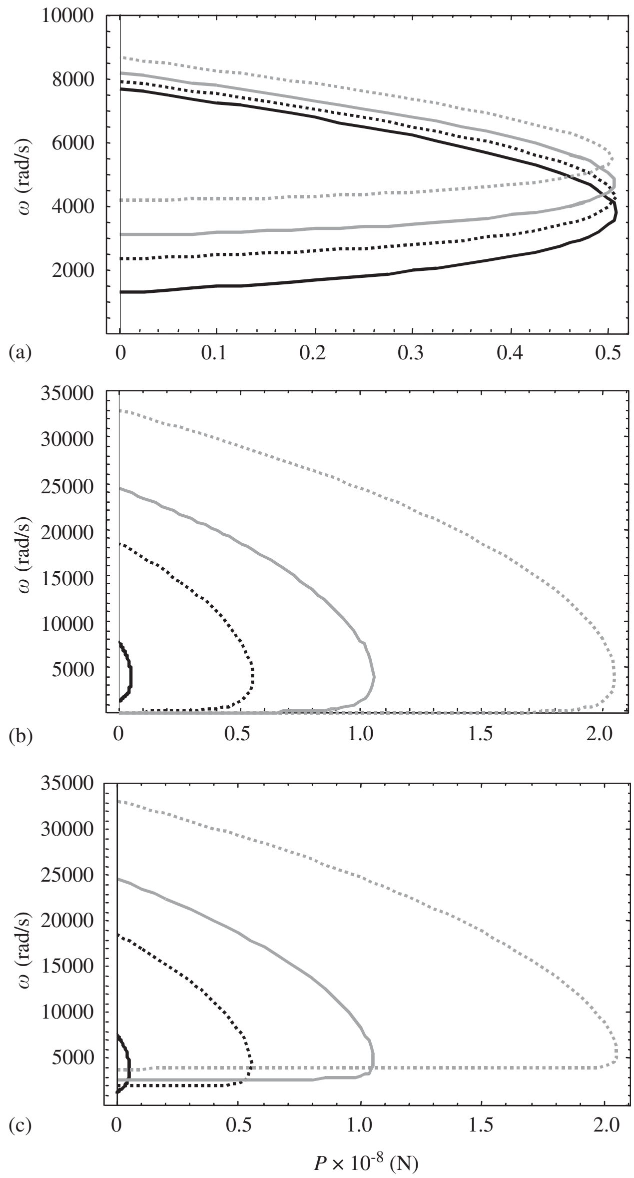

Fig. 10. Free vibration frequency versus concentrated load P for prismatic clamped–free beam resting on elastic foundation: (a) Winkler foundation—$K_1 = 0.5 \times 10^8\,\text{N/m}^2$ (.....), $P_{cr} = 5\,071\,794\,\text{N}$; $K_2 = 1.0 \times 10^8\,\text{N/m}^2$ (——), $P_{cr} = 5\,061\,370\,\text{N}$; $K_3 = 2.0 \times 10^8\,\text{N/m}^2$ (......), $P_{cr} = 5\,040\,529\,\text{N}$; (b) two parameter foundation at $K_1 = K_2 = K_3 = 0.0$, $C_1 = 0.5 \times 10^8\,\text{N}$, (.....), $P_{cr} = 55\,245\,768\,\text{N}$; $C_2 = 1.0 \times 10^8\,\text{N}$ (——), $P_{cr} = 105\,326\,413\,\text{N}$; $C_3 = 2.0 \times 10^8\,\text{N}$ (......), $P_{cr} = 205\,406\,409\,\text{N}$; (c) two parameter foundation at $K_1 = C_1 = 0.5 \times 10^8\,\text{N/m}^2$ (.....), $P_{cr} = 55\,234\,920\,\text{N}$; $K_2 = C_2 = 1.0 \times 10^8\,\text{N/m}^2$ (——), $P_{cr} = 105\,304\,289\,\text{N}$; $K_3 = C_3 = 2.0 \times 10^8\,\text{N/m}^2$; (......), $P_{cr} = 205\,361\,293\,\text{N}$. Line (——) represents solution with no elastic foundation taken into account at $P_{cr} = 5\,082\,219\,\text{N}$.

beam's parameters were the same as in the previously considered prismatic beam stability problem. The foundation parameters were: $K_1 = 0.5 \times 10^8\,\text{N/m}^2$, $K_2 = 1.0 \times 10^8\,\text{N/m}^2$, $K_3 = 2.0 \times 10^8\,\text{N/m}^2$, $C_1 = C_2 = C_3 = 0.0$ (the Winkler foundation) — the results in Fig. 10a; $K_1 = K_2 = K_3 = 0.0$, $C_1 = 0.5 \times 10^8\,\text{N}$, $C_2 = 1.0 \times 10^8\,\text{N}$, $C_3 = 2.0 \times 10^8\,\text{N}$ — the results in Fig. 10b and $K_1 = C_1 = 0.5 \times 10^8\,\text{N/m}^2$, $K_2 = C_2 = 1.0 \times 10^8\,\text{N/m}^2$, $K_3 = C_3 = 2.0 \times 10^8\,\text{N/m}^2$ — the results in Fig. 10c. In all the cases it was assumed that $\rho_F^V = 0$.

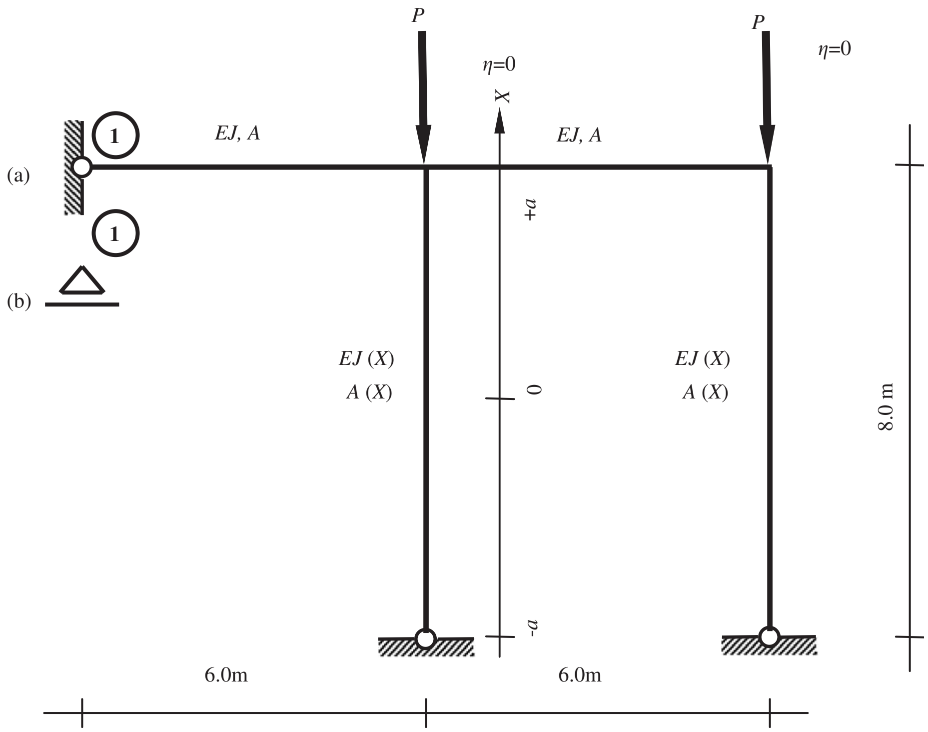

Fig. 11. Frame subjected to load P: (a) impossible horizontal translation of node '1' and (b) possible horizontal translation of node '1'.

4.3. Example 3

In order to demonstrate the applicability of the method to the analysis of more complex beam systems, the problem of stability of frame systems subjected to an axial potential load was solved. The schemes of the frames are shown in Figs. 11a and b. The first case was taken from monograph [17]. The other cases had been considered in paper [2] by the author. But in both works, the frames consist of Euler beams. In the present case, the axial deformability of the beams is taken into account.

The systems were analysed for the following material parameters: modulus of elasticity $E = 2.8 \times 10^{10}\,\mathrm{N/m^2}$, modulus of rigidity $G = 5E/12$ and density $\rho_{BV} = 2400\,\mathrm{kg/m^3}$. For each scheme in Fig. 11 two frames differing in the transverse dimensions of their beams were considered. The frames' cross sections were so selected that one frame was made up of slender beams while the other was made up of thick beams. The beams' geometrical parameters were as follows:

(a) The frame with slender beams: The spandrel beam rectangular in cross section, with cross-sectional area $A = 0.24\,\mathrm{m}^2$, the cross section's moment of inertia $J = 7.2 \times 10^{-3}\,\mathrm{m}^4$, shape factor $\kappa = 2/3$, the columns circular in cross section,

$$A(x) = \frac{\pi 10^{-2}}{8}(3 + x/a)^2\,\mathrm{m}^2, \quad J(x) = \frac{\pi 10^{-4}}{64}(3 + x/a)^4\,\mathrm{m}^4, \qquad \kappa = 3/4.$$

(b) The frame with thick beams: The spandrel beam rectangular in cross section, with cross-sectional area $A = 0.96\,\mathrm{m}^2$, the cross section's moment of inertia $J = 1.152 \times 10^{-1}\,\mathrm{m}^4$, shape factor $\kappa = 2/3$, the columns circular in cross section,

$$A(x) = 2\pi 10^{-2}(3 + x/a)^2\,\mathrm{m}^2, \quad J(x) = 4\pi 10^{-4}(3 + x/a)^4\,\mathrm{m}^4, \qquad \kappa = 3/4.$$

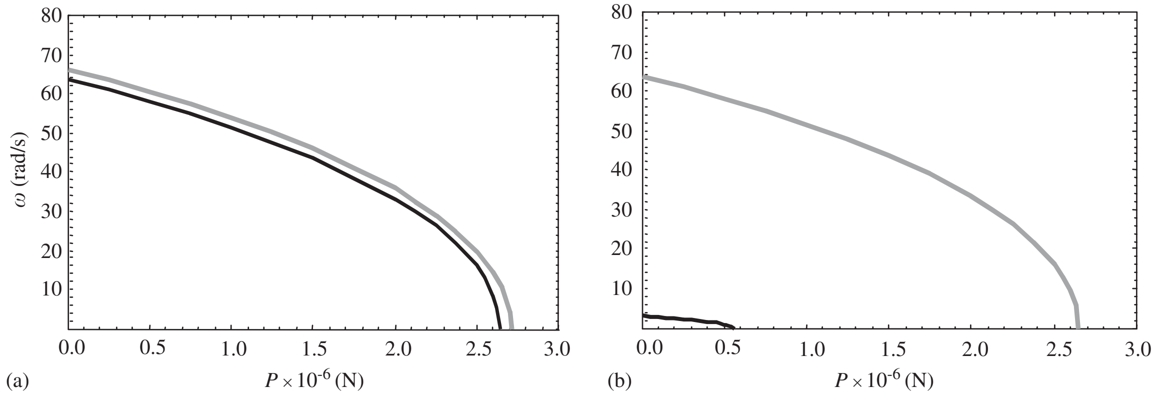

Fig. 12. Two first free vibration frequencies (Example 3a): (a) for frame (scheme shown in Fig. 11a) versus axial potential force P. Obtained values: $P_{1,\mathrm{cr}} = 2\,641\,321\,\mathrm{N}, P_{2,\mathrm{cr}} = 2\,710\,418\,\mathrm{N}$, (b) for frame (scheme shown in Fig. 11b) versus axial potential force P. Obtained values: $P_{1,\mathrm{cr}} = 557\,323\,\mathrm{N}$, $P_{2,\mathrm{cr}} = 2\,648\,051\,\mathrm{N}$.

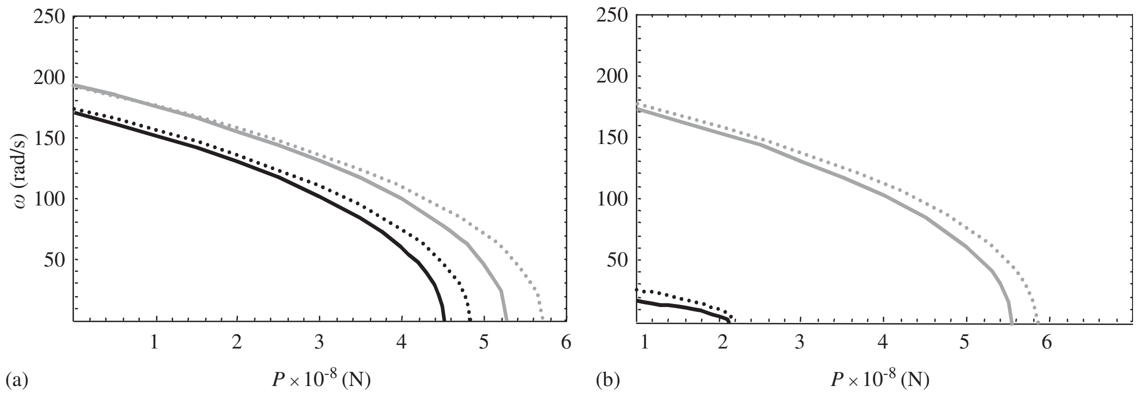

Fig. 13. Two first free vibration frequencies for Timoshenko (T) and Euler (E) model. (Example 3b): (a) for frame (scheme shown in Fig. 11a) versus axial potential force P. Obtained values: $P_{1,\mathrm{cr}} = 451\,629\,604\,\mathrm{N}$, $P_{2,\mathrm{cr}} = 526\,276\,923\,\mathrm{N}$, $P^E_{1,\mathrm{cr}} = 480\,909\,512\,\mathrm{N}$, $P^E_{2,\mathrm{cr}} = 570\,903\,811\,\mathrm{N}$. (b) For frame (scheme shown in Fig. 11b) versus axial potential force P. Obtained values: $P_{1,\mathrm{cr}} = 113\,666\,324\,\mathrm{N}$, $P_{2,\mathrm{cr}} = 454\,462\,931\,\mathrm{N}$, $P^E_{1,\mathrm{cr}} = 117\,274\,307\,\mathrm{N}$, $P^E_{2,\mathrm{cr}} = 484\,023\,457\,\mathrm{N}$. Symbols: ω_1—model T (——), model E (.....), ω_2—model T (——), model E (......).

The beam lengths and the local coordinate system used to describe the variable parameters of the columns are shown in Fig. 11. When solving the system, the equality of cross section rotation angles ϕ in the joints was assumed and 26 ($m = 25$) Chebyshev series terms were used to approximate displacements w and angle of rotation ϕ. The results are shown in Figs. 12 and 13.

5. Discussion

In the eigenforms for the simple supported beam one can distinguish groups of eigenforms characteristic of the Timoshenko beam. The parameters used in another (equivalent) way of defining the Timoshenko model, where it is assumed that the transverse displacements of the beam are the sum of the effects of bending and nondilatational strains and are expressed as $W(X,t) = W_M(X,t) + W_Q(X,t)$, will be helpful in describing the groups. Angular displacement Φ of the cross section and nondilatational deformation angle β, which are used in this paper to describe the model, are related to $W_M(X,t)$, $W_Q(X,t)$ as follows: $\Phi = \partial W_M/\partial X$, $\beta = \partial W_Q/\partial X$. Using the introduced displacements $W_M(X,t)$, $W_Q(X,t)$ one can describe the groups of eigenvalues in the solution for the simple supported beam. The first group comprises eigenforms with flexural and nondilatational displacements having the same sign, i.e. eigenforms $(W_1, \Phi_1) - (W_7, \Phi_7)$. The second group includes eigenforms whose displacements have opposite signs, i.e. $W_M(X,t)$, $W_Q(X,t)$. The first eigenform in this group is (W_9, Φ_9). Purely nondilatational eigenform (W_8, Φ_8) ($W \equiv 0,\ \ \Phi = -\beta = \text{const} \neq 0$) occurs between the two groups.

The natural frequencies calculated by the proposed method and the analytically determined exact frequencies were found to be in very good agreement. The relative error for the first nine free vibration frequencies does not exceed 3.7×10^{-5} for approximation with 20 polynomials and 9.2×10^{-8} for approximation with 30 polynomials. The eigenforms are similarly well approximated, but the cross section rotation function is usually less accurately approximated than the displacement function. The relative error for the first nine eigenforms is below 5.1×10^{-4} and 7.1×10^{-3} for, respectively, the displacement function and the rotation function for approximation with 20 polynomials and, respectively, 3.5×10^{-10} and 4.1×10^{-9} for approximation with 30 polynomials. If for the different methods of solving the eigenvalue problem one analyses the eigenvalues (Table 5) and the graphs of the relative errors defined by formula (32) (Fig. 4), it becomes apparent that the method proposed in this paper yields the most accurate results. As regards eigenvalues, the error is 10^5–10^{10} times smaller than in the case of the other methods. The approximation method using conventional power series has a similar accuracy, but only for the first three eigenvalues and eigenforms. From the fifth frequency onwards one gets incomparable complex solutions. It follows from Fig. 4 that for the proposed method the errors for the first eigenforms are only due to rounding errors. Similar conclusions can be drawn from the results (not reported here because of this paper's limited length) for the clamped–pin supported beam. Also in this case, the proposed method yields results with errors from several to about 10 or 20 orders of magnitude smaller than those of the other methods. The simply supported beam requires more comment: the best approximation was obtained using conventional Fourier series (the proposed method yielded results with errors larger by a few orders of magnitude), but it is rather an exception to the rule since such a good approximation by means of conventional Fourier series for the simple supported beam was due to the fact that the exact analytical solutions for this type of beam have the form of a finite linear combination of trigonometric functions. The same applies to the simply supported Euler beam for which the exact solutions have the form of sinusoids. In this case, approximation by means of just one (proper) Fourier series element yields an exact solution.

In the second example, the solutions of the stability problem (the free vibration frequency versus tangential compressive force diagrams shown in Figs. 7 and 9) for the Timoshenko beams and those for the Euler beams are compared. Since the ratio between the beams' transverse dimension and length was $\frac{1}{10}$, the results obtained for the Euler model should not differ considerably from the ones calculated for the Timoshenko beam. This is confirmed in Figs. 6 and 8. Even better agreement is obtained for the systems shown in Figs. 7 and 9. In the latter case, the differences were so slight that the graphs practically coincided and therefore they were not presented.

From the results obtained for the beam resting on the elastic foundation one can draw the following conclusions:

(a) By increasing the Winkler foundation's rigidity — parameter K (Fig. 10a) — one obtains (in the considered parameter range) only a slight reduction of P_{cr} which assumes the following successive values: $P_{cr} = 5\,071\,794,\ 5\,061\,370,\ 5\,040\,529\,\text{N}$. This conclusion confirms the results (for the problem of stability of a prismatic Timoshenko beam resting on the Winkler foundation) reported earlier by other authors, e.g. by Lee and Yang [19]. It becomes apparent that parameter K contributes more to an increase in the beam's first free vibration frequency than to an increase in its second frequency.
(b) If the value of parameter C in the foundation is increased at $K = 0$ (Fig. 10a), this results in a substantial increase in the critical load ($P_{cr} = 55\,245\,768,\ 105\,326\,413,\ 205\,406\,409\,\text{N}$) and in the second free vibration frequency while the first free vibration frequency significantly decreases.
(c) If the values of parameters C and K are simultaneously increased (in the tested parameter range), this results in an increase in the critical load ($P_{cr} = 55\,234\,920, 105\,304\,289, 205\,361\,293\,\text{N}$) and in the first and second free vibr ation frequency; the increase of the second frequency being much larger than that of the first frequency.

Example 3 shows that the proposed method can be applied to much more complex cases. Similarly as in Example 2, the slenderness ratio of the beams in the frame considered in Example 3a was sufficiently high to ensure that the results obtained using the Timoshenko model did not differ considerably from the ones obtained for the Euler beams — as shown by the comparison of the results (see Fig. 12) with the ones presented in an earlier paper by the author [2]. In both cases the graphs practically coincide. In Example 3b, where the frame is made up of thick beams, the differences are larger (see Fig. 13), particularly in the frame shown in Fig. 11a (see the graphs in Fig. 13a).

Appendix A

In this paper, in order to solve the system of differential equations a generalization of the following theorem concerning ordinary differential equations [16, p. 231] is used:

Theorem. *If function f satisfies a linear equation with order* $n>0$:

$$\sum_{m=0}^{n} \hat{P}_m(x) f^{(n-m)}(x) = \hat{P}(x) \tag{A.1}$$

and

$$Q_m(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{n-j}{m-j} \hat{P}_j^{(m-j)}(x), \quad m = 0, 1, \ldots, n, \tag{A.2}$$

where $\binom{n}{m} = n!/(m!(n-m)!)$ *and functions* $(Q_0 f)^{(n)}, (Q_1 f)^{(n-1)}, \ldots, Q_n f, \hat{P}$ *have determinable coefficients of the Chebyshev series, then for each integer k the following identity holds:*

$$\sum_{m=0}^{n} 2^{n-m} \sum_{j=0}^{m} b_{nmj}(k) a_{k-m+2j}[Q_m(x) f(x)] = \sum_{j=0}^{n} b_{nnj}(k) a_{k-n+2j}[\hat{P}(x)], \tag{A.3}$$

where $b_{nmj}(k)$ *are polynomials of integer k*

$$b_{nmj}(k) = (-1)^j \binom{m}{j} (k-n)_{n-m+j} (k-m+2j)(k+j+1)_{n-j} (k^2-n^2)^{-1}, \quad m = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, m. \tag{A.4}$$

$$(k)_n = \begin{cases} 1 & \text{for } n = 0, \\ k(k+1)(k+2)\ldots(k+n-1) & \text{for } n = 1, 2, 3, \ldots \end{cases} \tag{A.5}$$

and $a_k[h]$ *is the kth coefficient of expansion of function* $h(x)$ *into a Chebyshev series relative to Chebyshev polynomials of the first kind (the proof of this theorem can be found in Ref.* [16, pp. 231–234].

The generalization of the theorem consists in the transference of the differential equation approximate solution method (described by the theorem) onto systems of linear differential equations (see Ref. [16, p. 323]). In such a case, system of N equations can be presented in this matrix form

$$\sum_{m=0}^{n} \hat{\mathbf{P}}_\mathbf{m}(x) \mathbf{f}^{(\mathbf{n-m})}(x) = \hat{\mathbf{P}}(x), \tag{A.6}$$

where coefficients $\hat{\mathbf{P}}_\mathbf{m}(x)$ are square matrices of degree N and $\mathbf{f}(x)$ and $\hat{\mathbf{P}}(x)$ are N-element vectors. The differentiation of the vector means the differentiation of each of its components. If vector function $\mathbf{f}(x)$ satisfies system of Eqs. (A.6) and the theorem's assumptions hold good, then for each integer k the following identity is true:

$$\sum_{m=0}^{n} 2^{n-m} \sum_{j=0}^{m} b_{nmj}(k) a_{k-m+2j}[\mathbf{Q}_\mathbf{m}(x) \mathbf{f}(x)] = \sum_{j=0}^{n} b_{nnj}(k) a_{k-n+2j}[\hat{\mathbf{P}}(x)]. \tag{A.7}$$

Functions $\mathbf{Q_m}(x)$ in the formula are matrix equivalents of the functions defined by formula (A.2)

$$\mathbf{Q_m}(x) = \sum_{j=0}^{m} (-1)^{m+j} \begin{pmatrix} n-j \\ m-j \end{pmatrix} \hat{\mathbf{P}}_j^{(m-j)}(x), \quad m = 0, 1, \ldots, n \tag{A.8}$$

and $a_l[\mathbf{Q_m}(x)\ \mathbf{f}(x)]$ stands for a vector whose elements are the lth coefficients of the Chebyshev expansion of the components of vector $\mathbf{Q_m}(x)\ \mathbf{f}(x)$.

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